1

Dynamical Systems: A Tutorial

Mukul Majumdar and Tapan Mitra

1. Introduction

A dynamical system is described by a pair (X, f) where X is a nonempty set (called the *state space*) and f is a function (called the *law of motion*) from X into X. Thus, if x_t is the state of the system in period t, then

$$x_{t+1} = f(x_t) \tag{1.1}$$

is the state of the system in period $t + 1$.

In this chapter we always assume that the state space X *is a (nonempty) metric space (the metric is denoted by d) and* the law of motion f *is a continuous function from* X *into* X.

Once the initial state x (i.e., the state in period 0) is specified, we write $f^0(x) \equiv x$, $f^1(x) = f(x)$, and for every positive integer $j \geq 1$

$$f^{j+1}(x) = f(f^j(x)) \tag{1.2}$$

We refer to f^j as the *j-th iterate* of f. From any initial x, the *trajectory* from x is the sequence $\tau(x) = \{f^j(x)_{j=0}^{\infty}\}$. The *orbit* from x is the set $\gamma(x) = \{y : y = f^j(x) \text{ for some } j \geq 0\}$. The long run or asymptotic behavior of a trajectory $\tau(x)$ is described by its limit set $\omega(x)$ defined as the set of all limit points of $\tau(x)$.

Example 1.1: Let $X = \Re, f(x) = -x$. Verify that:
$\tau(1) = (1, -1, 1, -1, 1...)$
$\gamma(1) = \{1, -1\}$
$\tau(0) = (0, 0, 0...)$
$\gamma(0) = \{0\}$

Example 1.2: $X = \Re$, $f(x) = x + 1$. Here, $f^j(x) = x + j$ for all $j \geq 1$.

2. Fixed Points and Periodic Points

A point $x \in X$ is a *fixed point* if $x = f(x)$. A fixed point is often called a *steady state* or a *rest point* or an *equilibrium* of the dynamical system.

Exercise 2.1: Show that in Example 1.1, the only fixed point of f is the point 0. In Example 1.2, the function does not have a fixed point.

Example 2.1: Let $X = \Re$, $f(x) = x^2$. If x is any fixed point of f, it must satisfy the equation $x = x^2$. This means that the only fixed points of f are $0, 1$. More generally, consider the class of functions $f_c(x) = x^2 + c$, where c is a real number. For $c > 1/4$, f_c does not have any fixed point; for $c = 1/4$, f_c has a unique fixed point $x = 1/2$; for $c < 1/4$, f_c has a pair of fixed points.

A point $x \in X$ is a *periodic point* of *period* $k \geq 2$, if $f^k(x) = x$ and $f^j(x) \neq x$ for $1 \leq j < k$. Thus, to prove that x is a periodic point of period, say, 3, one must prove that x is a fixed point of f^3 and that it is *not* a fixed point of f and f^2. *Some writers consider a fixed point as a periodic point of period one.*

Example 2.2: Consider $X = \Re$, $f(x) = x^2 - 1$. The fixed points are $[\sqrt{5} + 1]/2$ and $[\sqrt{5} - 1]/2$. Note that $f(0) = -1$; $f(-1) = 0$. Hence, both 0 and -1 are periodic points of period 2.

$\tau(0) = (0, -1, 0, -1, ...)$
$\tau(-1) = (-1, 0, -1, 0, ...)$
$\gamma(-1) = \{-1, 0\}$
$\gamma(0) = \{0, -1\}$

More generally, consider $f_c(x) = x^2 + c$.

Here, $f_c^2(x) = (x^2 + c)^2 + c = x^4 + 2x^2c + c^2 + c$. Hence, to compute periodic points of period 2 one has to solve the equation:

$$x^4 + 2x^2c + c^2 + c = x \tag{2.1}$$

Observe that if f_c does have any fixed points, then these will also solve the equation.

Exercise 2.2: Consider the dynamical system in Example 1.2. Show that there is no periodic point.

Exercise 2.3: Let $X = [0,1]$. Consider the "tent map" defined by:

$$f(x) = \begin{cases} 2x & \text{for } x \in [0,1/2] \\ 2(1-x) & \text{for } x \in [1/2,1] \end{cases}$$

Note that f has two fixed points '0' and '2/3'. It is tedious to write out the functional form of f^2:

$$f^2(x) = \begin{cases} 4x & \text{for } x \in [0,1/4] \\ 2(1-2x) & \text{for } x \in [1/4,1/2] \\ 2(2x-1) & \text{for } x \in [1/2,3/4] \\ 4(1-x) & \text{for } x \in [3/4,1] \end{cases}$$

Verify the following:

(i) There are two periodic points of period 2, namely '2/5'and '4/5'.

(ii) There are three periodic points of period 3, namely '2/9', '4/9', '8/9'. It follows from a well-known theorem (see below) that there are periodic points of *all* periods

Let $\wp(X)$ be the set of all periodic points of X. We write $\aleph(X)$ to denote the set of non-periodic points.

We now note some useful results on the existence of fixed points of f.

Proposition 2.1: *Let $X = \Re$, and f be continuous. If there is a (nondegenerate) closed interval $I = [a,b]$ such that*

(i) $f(I) \subset I$ *or* (ii) $f(I) \supset I$

then there is a fixed point of f in I

Proof (sketch): (i) If $f(I) \subset I$, then $f(a) \geq a$, and $f(b) \leq b$. If $f(a) = a$ or $f(b) = b$, the conclusion is immediate. Otherwise, $f(a) > a$ and $f(b) < b$. This means that the function $g(x) = f(x) - x$ is positive at a and negative at b. Using the intermediate value theorem, $g(x^*) = 0$ for some x^* in (a,b). Then $f(x^*) = x^*$. (ii) If $f(I) \supset I$, by the Weierestrass' theorem, there are points

x_m and x_M in I such that $f(x_m) \leq f(x) \leq f(x_M)$ for all x in I. Write $f(x_m) = m$ and $f(x_M) = M$. Again, by the intermediate value theorem, $f(I) = [m, M]$. Since $f(I)$ is assumed to contain I, $m \leq a \leq b \leq M$. In other words,

$$f(x_m) = m \leq a \leq x_m,$$

and

$$f(x_M) = M \geq b \geq x_M.$$

The proof can now be completed by an argument similar to that in case (i).

Exercise 2.4: Let $X = [a, b]$, and f is a continuous function from X into X. Suppose that for all x in (a, b), the derivative $f'(x)$ exists and $|f'(x)| < 1$ Then f has a unique fixed point in X.

Proposition 2.2: *Let X be a nonempty compact convex subset of $\Re^n$, and f be continuous. Then there is a fixed point of f.*

A function $f : X \to X$ is a *contraction* if there is some β, $0 < \beta < 1$, such that for all $x, y \in X$, $x \neq y$, one has

$$d(f(x),\ f(y)) < \beta d(x, y)$$

If f is a contraction, f is continuous on X.

Proposition 2.3: *Let (X, d) be a (nonempty) complete metric space, and $f : X \to X$ a contraction. Then f has a unique fixed point $x^* \in X$. Moreover, for any x in X, the trajectory $\tau(x) = \{f^j(x)_{j=0}^{\infty}\}$ converges to x^*.*

Proof (sketch): Choose an arbitrary $x \in X$. Consider the trajectory $\tau(x) = (x_t)$ from x, where

$$x_{t+1} = f(x_t). \tag{2.2}$$

Note that $d(x_2, x_1) = d(f(x_1),\ f(x)) < \beta d(x_1, x)$; hence, for any $t \geq 1$,

$$d(x_{t+1}, x_t) < \beta^t d(x_1, x) \tag{2.3}$$

We note that

$$\begin{aligned} d(x_{t+2}, x_t) &\leq d(x_{t+2}, x_{t+1}) + d(x_{t+1}, x_t) \\ &< \beta^{t+1} d(x_1, x) + \beta^t d(x_1, x) \\ &= \beta^t (1+\beta) d(x_1, x) \end{aligned}$$

It follows that, for any integer $k \geq 1$,

$$d(x_{t+k}, x_t) < [\beta^t/(1-\beta)] d(x_1, x)$$

and this implies that (x_t) is a Cauchy sequence. Since X is assumed to be complete, $\lim_{t\to\infty} x_t = x^*$ exists. By continuity of f, and (2.2),

$$f(x^*) = x^*$$

If there are two distinct fixed points x^* and x^{**} of f, we see that there is a contradiction:

$$0 < d(x^*, x^{**}) = d(f(x^*), f(x^{**})) < \beta d(x^*, x^{**}) \tag{2.4}$$

where $0 < \beta < 1$ Q.E.D.

A fixed point x^* of f is *locally stable* (locally attracting) if there is an open set U containing x^* such that for all $x \in U$, the trajectory $\tau(x)$ converges to x^*.

A fixed point x^* is *repelling* if there is an open set U containing x^* such that for any $x \in U$, $x \neq x^*$, there is some $k \geq 1$, $f^k(x) \notin U$.

Consider a dynamical system (X, f) where X is a (nondegenerate) closed interval $[a, b]$, and f is continuous on $[a, b]$. Suppose that f is also continuously differentiable on (a, b). A fixed point $x^* \in (a, b)$ is *hyperbolic* if $|f'(x^*)| \neq 1$.

Proposition 2.4: *Let $X = [a, b]$ and f be continuous on $[a, b]$ and continuously differentiable on (a, b).*

(a) *If $x^* \in (a, b)$ is a hyperbolic fixed point of f and $|f'(x^*)| < 1$, then x^* is locally stable.*

(b) *If $x^* \in (a, b)$ is a hyperbolic fixed point with $|f'(x^*)| > 1$, then x^* is repelling.*

Proof: (a) There is some $\mu > 0$ such that $|f'(x)| < \beta < 1$ for all x in $I = [x^* - \mu, x^* + \mu]$. By the mean value theorem,

$$|x - x^*| = |f(x) - f(x^*) \leq \beta|x - x^*| < \beta\mu < \mu.$$

Hence, f maps I into I and, by the mean value theorem is a contraction on I. The result follows from Proposition 2.3.

(b) Exercise. Q.E.D.

We can define "a hyperbolic periodic point of period k" and define locally attracting and repelling periodic points accordingly.

Example 2.3: Going back to Example 2.2, consider once again the map $f(x) = x^2 - 1$. f^2 has four fixed points: the fixed points of f and $0, -1$. Both 0 and -1 are periodic points of period 2. Since

$$f^2(x) = x^4 - 2x^2$$

we see that the derivative of f^2 with respect to x, denoted by $[f^2(x)]'$, is given by

$$[f^2(x)]' = 4x^3 - 4x$$

Now, $[f^2(x)]'_{x=0} = [f^2(x)]'_{x=1} = 0$. Hence, both 0 and 1 are attracting fixed points of f^2.

Let x_0 be a periodic point of periods 2 and $x_1 = f(x_0)$. By definition $x_0 = f(x_1) = f^2(x_0)$ and $x_1 = f(x_0) = f^2(x_1)$. Now if f is differentiable, by the chain rule,

$$[f^2(x_0)]' = f'(x_1) \bullet f'(x_0)$$

More generally, suppose that x_0 is a periodic point of period k and its orbit is denoted by $\{x_0, x_1, ..., x_{k-1}\}$. Then, $[f^k(x_0)]' = f'(x_{k-1})...f'(x_0)$. It follows that $[f^k(x_0)]' = [f^k(x_1)]' = ...$ $[f^k(x_{k-1})]'$. We can now extend Proposition 2.4 appropriately.

Exercise 2.5: By using the graphs, if necessary, verify that the fixed and periodic points of the tent map in Exercise 2.3 are repelling.

While the contraction property of f ensures that the trajectories enter any neighborhood of the fixed point, there are examples of simple nonlinear dynamical systems in which trajectories "wander around" the state space. We shall now examine this feature more formally.

Proposition 2.5: *Let* $\{A_j\}_{j=0}^{\infty}$ *be a sequence of nonempty compact sets in* X *such that*

$$f(A_j) \supseteq A_{j+1}. \tag{2.5}$$

Then there is a nonempty compact set $Q \subset A_0$ *such that for all* $x \in Q$,

$$f^j(x) \in A_j \text{ for all } j. \tag{2.6}$$

Proof: Write $Q_0 = A_0$. Let g_j be the restriction of f to A_j. Define $Q_1 = (g_0)^{-1}(A_1)$. Since Q_1 is a closed subset of a compact set A_0, it is itself compact. It is nonempty. Construct in this manner a sequence of nonempty compact sets $Q_2 = (g_0)^{-1}$ $(g_1)^{-1}(A_2), ..., Q_{j+1} = (g_0)^{-1}...(g_j)^{-1}(A_{j+1})$. Then the sequence of nonempty compact sets $\{Q_j\}$ is "nested": $Q_{j+1} \subset Q_j$. Hence, $Q = \bigcap_{j=0}^{\infty} Q_j$ is nonempty and has the desired property by construction. Q.E.D.

Let us reflect on some of the implications of Proposition 2.5 by using the "tent map" of Exercise 2.3 for the sake of concreteness.

Exercise 2.3 (continued): Note that for the tent map both the intervals $H = [0, 1/2]$ and $T = [1/2, 1]$ have the property that $f(H) = f(T) = [0, 1]$. Let Ω be the uncountable set of all infinite sequences with two symbols H and T.

Choose an *arbitrary* element s of Ω This chosen s can be interpreted as an arrangement of a sequence $\{A_i\}$ of nonempty compact sets, where each A_i is either H or T. Let us stress informally that s can be viewed as a record of the outcomes of an infinite sequence of coin-tossings: the order in which H and T appear is not 'controlled' in any way whatsoever. Proposition 2.5 asserts that there is some x in $[0,1]$ from which the trajectory $\tau(x) = (f^j(x))$ satisfies $f^j(x) \in A_j$. See Saari (1991) for an elaboration of this theme.

3. Li-Yorke Chaos and the Sarkovskii Theorem

Here we take the state space X as a (nondegenerate) interval in the real line, (and f a continuous function from X into X). An interval is compact if it is closed and bounded. A subinterval of an interval I is an interval contained in I. Since f is continuous $f(I)$ is an interval. If I is a compact interval, so is $f(I)$. Suppose that a dynamical system (X, f) has a periodic point of period k. Can we conclude that it also has a periodic point of some other period $k' \neq k$? It is useful to look at a simple example first.

Example 3.1: Suppose that (X, f) has a periodic point of period $k (\geqq 2)$. Then it has a fixed point, (i.e., a periodic point of period one). To see this, consider the orbit γ of the periodic point of period k, and let us write

$$\gamma = \{x^{(1)}, ..., x^{(k)}\}$$

where $x^{(1)} < x^{(2)} < ... < x^{(k)}$. Both $f(x^{(1)})$ and $f(x^{(k)})$ must be in γ. This means that

$$f(x^{(1)}) = x^{(i)} \text{ for some } i > 1$$

and

$$f(x^{(k)}) = x^{(j)} \text{ for some } j < k.$$

Hence, $f(x^{(1)}) - x^{(1)} > 0$ *and* $f(x^{(k)}) - x^{(k)} < 0$. By the intermediate value theorem, there is some x in X such that $f(x) = x$.

We shall now state the Li-Yorke theorem and provide a brief sketch of the proof.

Proposition 3.1: *Let I be an interval and $f : I \to I$ be continuous. Assume that there is some point a in I for which there are points $b = f(a)$, $c = f(b)$ and $d = f(c)$ satisfying:*

$$d \leq a < b < c \ (or,\ d \geq a > b > c). \tag{3.1}$$

Then:

[T.1] *For every positive integer* $k = 1, 2, \ldots$ *there is a periodic point of period* k *in* I.

[T.2] (i) *There is an uncountable* S *of* $\aleph(X)$ *such that for all* p, q *in* S, $p \neq q$,

$$\limsup_{n\to\infty} |f^n(p) - f^n(q)| > 0; \tag{3.2}$$

$$\liminf_{n\to\infty} |f^n(p) - f^n(q)| = 0. \tag{3.3}$$

(ii) *If* $p \in S$, *and* $q \in \wp(X)$

$$\limsup_{n\to\infty} |f^n(p) - f^n(q)| > 0$$

Proof of [T.1] [sketch]:

Step 1: Let G be a real-valued continuous function on an interval I. For any compact subinterval I_1 of $G(I)$ there is a compact subinterval Q of I such that $G(Q) = I_1$.

Proof of Step 1: One can figure out the subinterval Q directly as follows. Let $I_1 = [G(x), G(y)]$ where x, y are in I. Assume that $x < y$. Let r be the last point of $[x, y]$ such that $G(r) = G(x)$; let s be the first point after r such that $G(s) = G(y)$. Then $Q = [r, s]$ is mapped onto I_1 under G. The case $x > y$ is similar. Q.E.D.

Step 2: Let I be an interval and $f : I \to I$ be continuous. Suppose that $(I_n)_{n=0}^{\infty}$ is a sequence of compact subintervals of I and, for all n,

$$I_{n+1} \subset f(I_n). \tag{3.4}$$

Then there is a sequence of compact subintervals (Q_n) of I such that, for all n,

$$Q_{n+1} \subset Q_n \subset Q_0 = I_0 \tag{3.5}$$

and

$$f^n(Q_n) = I_n. \tag{3.6}$$

Hence, there is

$$x \in \bigcap_n Q_n \text{ such that } f^n(x) \in I_n \text{ for all } n \tag{3.7}$$

Proof of Step 2: The construction of the sequence Q_n proceeds "inductively" as follows: Define $Q_0 = I_0$. Recall that f^0 is defined as the identity mapping, so $f^0(Q_0) = I_0$ and $I_1 \subset f(I_0)$. If Q_{n-1} is defined as a compact subinterval such that $f^{n-1}(Q_{n-1}) = I_{n-1}$, then $I_n \subset f(I_{n-1}) = f^n(Q_{n-1})$. Use Step 1, with $G = f^n$ on Q_{n-1} in order to get a compact subinterval Q_n of Q_{n-1} such that $f^n(Q_n) = I_n$. This completes the induction argument (establishing (3.5) and (3.6)). Compactness of Q_n leads to (3.7).

Now we prove [T.1]. Assume that $d \le a < b < c$ (the other case $d \ge a > b > c$ is treated similarly).

Write $K = [a, b]$ and $L = [b, c]$.
Let k be any positive integer.
For $k > 1$, define a sequence of intervals (I_n) as follows:

$I_n = L$ for $n = 0, 1, 2, ...k - 2$; $I_{k-1} = K$, and $I_{n+k} = I_n$ for $n = 0, 1, 2.....$

For $k = 1$, let $I_n = L$ for all n.

Let Q_n be the intervals in Step 2. Notice that $Q_k \subset Q_0 = I_0$ and $f^k(Q_k) = I_k = I_0$. Hence, Proposition 2.1 applied to f^k gives us a fixed point p_k of f^k in Q_k. Now, p_k cannot have period less than k; otherwise, we need to have $f^{k-1}(p_k) = b$, contrary to $f^{k+1}(p_k) \in L$. Q.E.D.

Proof of [T.2]: Let $\mathcal{M}$ be the set of sequences $M = \{M_n\}_{n=1}^{\infty}$ of intervals with

A1: $M_n = K$ or $M_n \subset L$, and $f(M_n) \supset M_{n+1}$
if $M_n = K$ then
A2: n is the square of an integer and $M_{n+1}, M_{n+2} \subset L$

Of course if n is the square of an integer, then $n+1$ and $n+2$ are not, so the last requirement in (A.2) is redundant. For $M \in \mathcal{M}$ let $P(M, n)$ denote the number of i's in $\{1, ..., n\}$ for which $M_i = K$. For each $r \in (3/4, 1)$ choose $M^r = \{M_n^r\}_{n=1}^{\infty}$ to be a sequence in $\mathcal{M}$ such that

A3: $\lim_{n\to\infty} P(M^r, n^2)/n = r.$

Let $\mathcal{M}_0 = \{M^r : r \in (3/4, 1)\} \subset \mathcal{M}$. Then $\mathcal{M}_0$ is uncountable since $M^{r_1} \neq M^{r_2}$ for $r_1 \neq r_2$. For each $M^r \in \mathcal{M}_0$, by Step 2, there exists a point x_r, with $f^n(x_r) \in M^r_n$ for all n.
Let $S = \{x_r : r \in (3/4, 1)\}$. Then S is also uncountable. For $x \in S$, let $P(x, n)$ denote the number of i's in $\{1, ..., n\}$ for which $f^i(x) \in K$. We can never have $f^k(x_r) = b$, because then x_r would eventually have period 3, contrary to (A.2). Consequently $P(x_r, n) = P(M^r, n)$ for all n, and so

$$\rho(x_r) = \lim_{n\to\infty} P(X_r, n^2) = r$$

for all r. We claim that

A4: for $p, q \in S$, with $p \neq q$, there exist finitely many n's such that $f^n(p) \in K$ and $f^n(q) \in L$ or vice versa.

We may assume $\rho(p) > \rho(q)$. Then $P(p, n) - P(q, n) \to x$, and so there must be infinitely many n's such that $f^n(p) \in K$ and $f^n(q) \in L$.
Since $f^2(b) = d \leqq a$ and f^2 is continuous, there exists $\delta > 0$ such that $f^2(x) < (b + d)/2$ for all $x \in [b - \delta, b] \subset K$. If $p \in S$ and $f^n(p) \in K$, then (A.2) implies $f^{n+1}(p) \in L$ and $f^{n+2}(p) \in L$. Therefore $f^n(p) < b - \delta$. If $f^n(q) \in L$, then $f^n(q) \geqq b$ so

$$|f^n(p) - f^n(q)| > \delta.$$

By claim (A.4), for any p, $q \in S$, $p \neq q$, it follows

$$\limsup_{n\to\infty} |f^n(p) - f^n(q)| \geqq \delta > 0$$

Hence (3.1) is proved. This technique may be similarly used to prove that [T.2(ii)] is satisfied.

Proof of (3.3): Since $f(b) = c$, $f(c) = d$, $d \leqq a$, we may choose intervals $[b^n, c^n]$, $n = 0, 1, 2, ...,$such that
(a) $[b, c] = [b^0, c^0] \supset [b^1, c^1] \supset ... \supset [b^n, c^n] \supset ...,$

(b) $f(x) \in (b^n, c^n)$ for all $x \in (b^{n+1}, c^{n+1})$,
(c) $f(b^{n+1}) = c^n$, $f(c^{n+1}) = b^n$.

Let $A = \bigcap_{n=0}^{\infty} [b^n, c^n]$, $b^* = \inf A$ and $c^* = \sup A$, then $f(b^*) = c^*$ and $f(c^*) = b^*$, because of (c).

In order to prove (3.3) we must be more specific in our choice of the sequences M^r. In addition to our previous requirements on $M \in \mathcal{M}$, we assume that if $M_k = K$ for both $k = n^2$ and $(n+1)^2$ then $M_k = [b^{2n-(2j-1)}, b^*]$ for $k = n^2 + (2j-1)$, $M_k = [c^*, c^{2n-2j}]$ for $k = n^2 + 2j$ where $j = 1, ..., n$. For the remaining k's which are not squares of integers, we assume $M_k = L$.

It is easy to check that these requirements are consistent with (A1) and (A2), and that we can still choose M^r so as to satisfy (A3). From the fact that $\rho(x)$ may be thought of as the limit of the fraction of n's for which $f^{n^2}(x) \in K$, it follows that for any r^*, $r \in (3/4, 1)$ there exist infinitely many n such that $M_k^r = M_k^{r^*} = K$ for both $k = n^2$ and $(n+1)^2$. To show (3.3), let $x_r \in S$ and $x_{r^*} \in S$. Since $b^n \to b^*$, $c^n \to c^*$ as $n \to \infty$, for any $\varepsilon > 0$ there exists N with $|b^n - b^*| < \varepsilon/2$, $|c^n - c^*| < \varepsilon/2$ for all $n > N$. Then, for any n with $n > N$ and $M_k^r = M_k^{r^*} = K$ for both $k = n^2$ and $(n+1)^2$, we have

$$f^{n^2+1}(x_r) \in M_k^r = [b^{2n-1}, b^*]$$

with $k = n^2 + 1$ and $f^{n^2+1}(x_r)$ and $f^{n^2+1}(x_{r^*})$ both belong to $[b^{2n-1}, b^*]$. Therefore, $|f^{n^2+1}(x_r) - f^{n^2+1}(x_{r^*})| \leqq \varepsilon$. Since there are infinitely many n with this property, $\liminf_{n\to\infty} |f^n(x_r) - f^n(x_{r^*})| = 0$. Q.E.D.

We shall now state Sarkovskii's theorem on periodic points. Consider the following Sarkovskii ordering of the positive integers:

$$\begin{aligned} &3 \rhd 5 \rhd 7 ... \rhd 2.3 \rhd 2.5 ... \rhd 2^2 3 \rhd 2^2 5 \rhd ... \\ &\rhd 2^3.3 \rhd 2^3.5 ... \rhd 2^3 \rhd 2^2 \rhd 1 \end{aligned} \qquad \text{(SO)}$$

In other words, first list all the odd integers beginning with 3; next list 2 times the odds, 2^2 times the odds, etc. Finally, list all the powers of 2 in decreasing order.

Proposition 3.2: *Let $X = \Re$, f be a continuous function from X into X. Supppose that f has a periodc point of period k. If $k \rhd l$ in the Sarkovskii ordering* (SO), *then f has a periodic point of period l.*

Proof: See Block, Guckenheimer, Misiurewicz, and Young (1980).

It follows that if f has only finitely many periodic points, then they all necessarily have periods which are powers of two.

Exercise 3.1: Show that if f has a periodic point of period 4 it has a periodic point of period 2. [Hint: Suppose that $\{x_1, x_2, x_3, x_4\}$ is the orbit of a periodic point of period 4. Let

$$x_1 < x_2 < x_3 < x_4$$

Choose a point a between x_2 and x_3. By considering all possible transitions among the points, one can show that there are two cases. The first case occurs if both $f(x_1) > a$ and $f(x_2) < a$. Then one must have $f(x_3) < a$ and $f(x_4) < a$. Let $I_0 = [x_1, x_2]$ and $I_1 = [x_3, x_4]$. Show that $f(I_1) \supset I_0$ and $f(I_0) \supset I_1$. It follows that f^2 has a fixed point, and it is in fact a periodic point of period 2.

The other case occurs when either one of x_1 or x_2 is mapped to the right of a but the other is not. For definiteness, suppose $f(x_1) > a$ and $f(x_2) < a$. Consequently, one has $f(x_2) = x_1$. Let $I_0 = [x_2, x_3]$ and $I_1 = [x_1, x_2]$. Then we have $f(I_0) \supset I_1$ and $f(I_1) \supset I_0 \cup I_1$. This is the same situation that occurs in the Li-Yorke theorem. Hence, there is a periodic point of period 2 (and, in fact, a periodic point of any period).

Example 3.1: If f has a periodic point of period $2^n (n \geq 3)$, it has a periodic point of period 2^m where $n > m$. Let $l = 2^{n-2}$ and consider $G(x) = f^l(x)$. A periodic point of period 2^n for f is a periodic point of period 4 of G. From Exercise 3.1 we know then that G has a periodic point of period 2. But this means that f has a periodic point of period 2^{n-1}.

Exercise 3.2: Let X be an interval and $f : X \to X$ be continuous and satisfy the following property: there are two subintervals A, B of X where $B = f(A)$ such that

(i) $A \cap B = \phi$

(ii) $f(B) \supset A \cup B$

Note that $f(B) \supset B$ implies that f has a fixed point. Also

$$f^2(A) \equiv f(f(A)) = f(B) \supset A$$

Hence, f^2 has a fixed point in A, i.e., there is some $y \in A$ such that $y = f^2(y)$. But $y \neq f(y)$ by Condition (i). Hence, f is a periodic point of period 2.

Show that (X, f) has periodic point of all periods. For a detailed discussion of such maps see Diamond (1976) and Day (1994).

4. Some Related Concepts

4.1 Devaney's Definition of Chaos

In his widely used text, Devaney (1986) introduced a definition of a chaotic dynamical system that captures several interesting properties that we first discuss.

a. Topological Transitivity

(X, f) is *topologically transitive* if for any pair of nonempty open sets U and V, there exists $k \geq 1$ such that $f^k(U) \cap V \neq \phi$.

Proposition 4.1: *If there is some x such that $\gamma(x)$, the orbit from x, is dense in X, then (X, f) is topologically transitive.*

Proposition 4.2: *Let X be a (nonempty) compact metric space. Assume that (X, f) is topologically transitive. Then there is some $x \in X$ such that the orbit $\gamma(x)$ from x is dense in X.*

Proof: Since X is compact, it has a countable base of open sets, i.e., there is a family $\{V_n\}$ of open sets in X with the property that if M is any open subset of X, there is some $V_n \subset M$.

Corresponding to each V_n, define the set O_n as follows:

$$O_n = \{x \in X : f^j(x) \in V_n, \quad \text{for some } j \geq 0\}$$

O_n is open, by continuity of f. By topological transitivity it is also dense in X. By Baire category theorem, the intersection $O = \cap_n O_n$ is also open and dense in X. Take any $x \in O$, and consider the orbit $\gamma(x)$ from x. Take any y in X and any open M containing y. Then M contains some V_n. Since x belongs to the corresponding O_n, there is some element of $\gamma(x)$ in V_n. Hence, $\gamma(x)$ is dense in X. Q.E.D.

b. Sensitivity to Initial Condition

A dynamical system (X, f) has *sensitive dependence on initial condition* if there is $\partial > 0$ such that, for any $x \in X$ and any neighborhood N of x there exist y and an integer $j \geq 0$ with the property $|f^j(x) - f^j(y)| > \partial$.

Devaney asserted that if a dynamical system "possesses sensitive dependence on initial condition, then for all practical purposes, the dynamics defy numerical computation. Small errors in computation which are introduced by round-off may become magnified upon iteration. The results of numerical computation of an orbit, no matter how accurate, may bear no resemblance whatsoever with the real orbit".

For a dynamical system (X, f) where the state space X is a subset of the real line, there is another concept of sensitive dependence explored by Guckenheimer (1979). Let us state it formally here for future reference. A dynamical system (X, f) has *Guckenheimer dependence on initial conditions* if there is a set of $Y \subset X$ of positive Lebesgue measure and an $\varepsilon > 0$ such that given any $x \in Y$ and any neighborhood U of x, there is $y \in Y$ and $n \geq 0$ such that $|f^n(x) - f^{(n)}(y)| > \varepsilon$.

c. A Chaotic Dynamical System

A dynamical system (X, f) is *chaotic in the sense of Devaney* if:

(i) (X, f) is topologically transitive.
(ii) (X, f) has sensitive dependence on initial condition.
(iii) The set of periodic points $\wp(X)$ of X is dense in X.

Exercise 4.1: Let X be a metric space (with an infinite number of elements) and $f : X \to X$ a continuous function. One can show that the properties (i) and (iii) imply (ii). See Banks et. al. (1992) for details.

Example 4.1: Let $X = [0,1]$ and $f(x) = 4x(1-x)$ is chaotic in Devaney's sense.

Example 4.2: Consider the space of all sequences of '0' and '1'; formally

$$\mathcal{S}_2 = \{s = (s_t) : s_t = 0 \text{ or } 1\}.$$

On $\mathcal{S}_2$ define a metric d by

$$d(s, s') = \sum_{t=0}^{\infty} |s_t - s'_t|/2^t$$

Note the following property:

R.4.1: Suppose that s and s' are in $\mathcal{S}_2$ and $s_t = s'_t$ all $t = 0, 1, ...n$. Then $d(s, s') \leq 1/2^n$; on the other hand, if $d(s, s') < 1/2^n$, then $s_t = s'_t$ for all $t = 0, ...n$.

Proof: Suppose that $s_t = s'_t$ for $t = 0, ..., n$. Then

$$d(s, s') = \sum_{t=0}^{n} \frac{|s_t - s'_t|}{2^t} + \sum_{t=n+1}^{\infty} \frac{|s_t - s'_t|}{2^t} \leq \frac{1}{2^n}$$

On the other hand, if $s_t \neq s'_t$ for k some $k \leq n$, then we must have

$$d(s, s') \geqq \frac{1}{2^k} \geq \frac{1}{2^n}$$

Consequently, $d(s, s') < 1/2^n$ implies $s_t = s'_t, t = 0, ..., n$. Q.E.D.

Define the "shift map" $\sigma : \mathcal{S}_2 \to \mathcal{S}_2$ as;

$$\sigma(s_0, s_1, ...) = (s_1, s_2, ...).$$

R.4.2: σ is continuous.

Proof: Let $\varepsilon > 0$ be given and $s = (s_0, s_1, ...)$. Choose n such that $1/2^n < \varepsilon$. Let $\delta = (1/2^{n+1})$. If $m = (m_0, m_1....)$ satisfies $d(s, m) < \delta$, then by R.4.1, we must have $s_t = m_t$ for $0 \leq t \leq n+1$. Hence, $d(\sigma(s), \sigma(m)) \leq 1/2^n < \varepsilon$, again by R.4.1. Q.E.D.

R.4.3: a) The set of periodic points of σ is dense in $\mathcal{S}_2$.
b) There is a dense orbit for σ in $\mathcal{S}_2$.

Proof: a) In order to prove that the set of periodic points is dense in $\mathcal{S}_2$, we have to show that if $s = (s_0, s_1, ..., s_t, ...)$ is an arbitrary element of $\mathcal{S}_2$, there is a sequence τ_n of periodic points in $\mathcal{S}_2$ that converges to s (in the metric d). This sequence is constructed as follows:

$$\tau_0 = (s_0, s_0, s_0, s_0, ...)$$
$$\tau_1 = (s_0, s_1; s_0, s_1; ...)$$

•

•

$$\tau_n = (s_0, s_1, ..., s_n; s_0, ..., s_n; , ...)$$

Then, by R.4.1., $d(s, \tau_n) \leq 1/2^n$.
(b) Consider

$$s^* = (0, 1| \quad 0, 0; 0, 1; 1, 0; 1, 1| \quad 0, 0, 0; 0, 0, 1.....)$$

The sequence s^* is constructed by listing all blocks of 0's and 1's of length n, then length $n+1$, etc. Clearly, some iterate of applied to s^* yields a sequence that agrees with any given sequence in an arbitrarily large number of places. Hence (b) follows (from an application of R.4.1). Q.E.D.

5. Ergodic Chaos

Let $(X, \Im, \mu)$ be a probability space: here X is a set, $\Im$ is a σ field of subsets of X, and μ is a probability measure on $\Im$. The elements of $\Im$ are *measurable subsets* of X.

Example 5.1: X is a metric space; $\Im$ is the Borel σ field of X; μ is any probability measure on X.

Example 5.2: $X = [0, 1]$; $\Im$ is the Borel σ field of X; m is the Lebesgue measure. Recall that for any subinterval (c, d) of $[0, 1]$, $m\{(c, d)\} = d - c$. Also, m is nonatomic, i.e., $m\{x\} = 0$ for any $x \in X$.

A mapping (transformation, function) f from X into X is *measurable* if, for any $S \in \Im$, $f^{-1}(S) \in \Im$.

Example 5.3: Let X be any metric space, $\Im$ its Borel σ field and $f : X \to X$ be continuous. Then f is measurable.

In what follows, any mapping is understood to be a measurable mapping.

A measurable mapping f is *measure preserving* and μ is *invariant* if $\mu(A) = \mu(f^{-1}(A))$.

If f is a measurable mapping on a probability space $(X, \Im, \mu)$ and E is any measurable subset of X, a point $x \in E$ is *recurrent* (with respect to E and f) if $f^n(x) \in E$ for at least one positive integer n.

Proposition 5.1: *If f is measure preserving, and $E \in \Im$, then (μ) almost every point of E is recurrent.*

Proof: If not then the set F of those points of E that never return to E is a set of positive measure. The set F is measurable since

$$F = E \cap f^{-1}(X - E) \cap f^{-2}(X - E) \cap \ldots$$

If $x \in F$, then none of the points $f(x)$, $f^2(x)$, $f^3(x)$, ... belongs to F, or, in other words, F is disjoint from $f^{-n}(F)$ for all positive n. It follows that F, $f^{-1}F$, $f^{-2}F$ are all pairwise disjoint, since

$$f^{-n}F \cap f^{-(n+k)}F = f^{-n}(F \cap f^{-k}F).$$

Since f is measure preserving, and the measure of X is one, this is a contradiction. Q.E.D.

The recurrence theorem implies a stronger version. Not only is it true that for almost every $x \in E$, at least one term of the sequence $f(x)$, $f^2(x)$,... belongs to E; in fact, for almost every x in E, there are infinitely many values of n such that $f^n(x) \in E$. The idea of the proof is to apply the recurrence theorem to each iterate of f. If F_n is the set of those points of E that never return to E under the action of f^n, then by the recurrence theorem $\mu(F_n) = 0$. If $x \in E - (F_1 \cup F_2 \cup ...)$, then $f^n(x) \in E$, for some positive n since $x \in E - F_1$. Since $x \in E - F_n$, it follows that $f^{kn}(x) \in E$ for some positive k. The strengthened version of the recurrence theorem follows.

We now turn to the concept of ergodicity. A set $E \in \Im$ is invariant under f if and only if $f^{-1}(E) = E$; this means that "$x \in E$" if and only if "$f(x) \in E$". A measurable function f is *ergodic* if and only if it has trivial invariant sets, i.e., $E \in \Im$ and $f^{-1}(E) = E$ implies that either $\mu(E) = 0$ or $\mu(E) = 1$. In this case we shall refer to μ as an *ergodic measure* (for f).

A major result in the theory of dynamical systems is the celebrated "ergodic theorem".

Proposition 5.2: *Let $(X, \Im, \mu)$ be a probability space and f be measure preserving and ergodic. Let $g(\bullet)$ be an integrable function. Then*

$$\lim_{n\to\infty} 1/n \sum_{i=0}^{n-1} g(f^i(x)) = \int_X g d\mu \quad \textit{for almost all } x \in X \tag{5.1}$$

For an extended discussion, see Halmos (1956).

An immediate implicaton of the ergodic theorem is the following: for any measurable E, and for μ-almost every x, $\mu(E)$ is the fraction of time that the trajectory from x spends in E.

Consider the case when X is a metric space, and $\Im$ is its Borel σ field. A probability measure μ is *strictly positive* if the μ measure of any nonempty open set is positive.

Proposition 5.3: *Let X be a separable metric space; $\Im$ its Borel σ field, and μ be a strictly positive probability measure. If f is any measure preserving, ergodic mapping from X into X, then for μ almost every $x \in X$, the orbit $\gamma(x)$ is dense in X.*

Proof: The orbit of x is not dense if and only if there is a nonempty basic open set G such that x belongs to the intersection of all $X - f^n G$. Since this intersection is an invariant set disjoint from G and $\mu(G) > 0$, it follows that it has μ measure zero. If x does not belong to any of this countable class of sets of measure zero (one for each basic open set), then x has a dense orbit. Q.E.D.

We now introduce the notion of the *support* of a measure.

Proposition 5.4: *Let X be a separable metric space, $\Im$ its Borel σ field and μ a proability measure on x. Then there exists a unique closed set C_μ satisfying* (i) $\mu(C_\mu) = 1$; (ii) *if D is any closed set such that $\mu(D) = 1$, then $C_\mu \subset D$. Moreover,* (iii) *C_μ is the set of all points $x \in X$ having the property that $\mu(U) > 0$ for each open set U containing x.*

Proof: See Parthasarathy (1967, p. 28).

The closed set C_μ in Proposition 5.3 is called the *support* of μ. Note that if the support of μ is the entire set X, then μ is necessarily strictly positive.

Exercise 5.1: Consider $X = [0, 1]$, and let m be the Lebesgue measure on the Borel σ field of x. Then m is necessarily strictly positive. Indeed, any probability measure equivalent to m is also strictly positive. Recall that μ and m are equivalent if "$\mu(E) = 0$ if and only if $m(E) = 0$".

Exercise 5.2: Let $X = [0, 1]$. Suppose that a probability measure v is given by its density function $i(\cdot)$.

$$v(E) = \int_E i(x) m(dx)$$

If the density function $i : [0, 1] \to [0, 1]$ is strictly positive, i.e., $i(x) > 0$ for all $x \in [0, 1]$, then v is strictly positive.

In what follows in this section, the state space X of the dynamical system (X, f) is taken to be a closed interval $[a, b]$.

A dynamical system (X, f) is said to exhibit *ergodic chaos* if there is an ergodic, invariant measure v which is absolutely continuous with respect to m, the Lebesgue measure.

An *attractor* for f is a closed subset F of X such that $\omega(x) = F$ for all x in a set of positive Lebesgue measure. We now state two results on the existence of ergodic chaos.

Proposition 5.5: *Let (X, f) be a dynamical system with $X = [a, b]$. Suppose that there is some $y \in (a, b)$ and f restricted to both (a, y) and (y, b) is*

(i) *strictly monotone;*

(ii) *twice continuously differentiable; and*

(iii) *there is $\lambda > 1$ such that $f'(x) \geq \lambda > 1$ for μ-almost every x in (a, y) and (y, b).*

Then there is a unique ergodic invariant measure μ which is absolutely continuous with respect to m, and the support of m is the attractor for m-almost every $x \in [a, b]$.

Proposition 5.6: *Let (X, f) be a dynamical system with $X = [a, b]$. Suppose that there is $x^* \in (a, b)$ such that*

(i) *f is twice continuously differentiable on $[a, b]$*

(ii) *$f'(x) > 0$ for $x < x^*$; $f'(x^*) = 0$; $f''(x^*) < 0$; $f'(x) < 0$ for $x > x^*$.*

(iii) *$f(x) > x$ for all $x \in (a, x^*)$; $f(x^*) \in (x^*, b)$; $\mathcal{S}f(x) \equiv [f'''(x)/f'(x)] - 3/2[f''(x)/f'(x)]^2 < 0$ for all $x \in X$ except $x = x^*$.*

(iv) *There exists* $K \geq 2$ *such that* $y = f^K(x^*)$ *is an unstable fixed point of* f.

Then f *exhibits ergodic chaos.*

For a more detailed discussion see Day and Pianigiani (1991) and Misiurewicz (1981).

6. The Quadratic Family

Let $X = [0,1]$ and $I = [1,4]$. The *quadratic family of maps* is then defined by

$$f_\mu(x) = \mu x(1-x) \text{ for } (x,\mu) \in X \times I$$

We interpret x as the *variable* and μ as the *parameter* of the map h.

A few observations about the quadratic family are useful at this point. Note that, for each parameter specification $\mu \in I$, the state space is the same. Thus we can conveniently examine a family of dynamical systems $(X, f_\mu(x)$ parametrized by μ.

For each $\mu \in I$, f_μ has exactly one *critical point* (i.e., a point where $f'_\mu(x) = 0$, and this critical point (equal to 0.5) is independent of the parameter μ.

6.1. Stable Periodic Orbits

Even though there may be an infinite number of periodic orbits for a given dynamical system (as in the Li-Yorke theorem), a striking result due to Julia and Singer, informs us that there can be *at most one stable* periodic orbit.

Proposition 6.1: *Let* $X = [0,1]$, $I = [1,4]$; *given some* $\overline{\mu} \in I$, *define* $f_{\overline{\mu}}(x) = \overline{\mu}x(1-x)$ *for* $x \in X$. *Then there can be at most one stable periodic orbit. Furthermore, if there is a stable periodic orbit, then* $\omega(0.5)$, *the limit set of* $x^* = 0.5$, *must coincide with this orbit.*

Suppose, now, that we have a stable periodic orbit. This means that the asymptotic behavior (limit sets) of trajectories from all initial states "near" this periodic orbit must coincide with the periodic orbit. But what about the asymptotic behavior of trajectories from other initial states? If one is interested in the behavior of a "typical" trajectory, a remarkable result, due to Misiurewicz (1983), settles this question. Recall that m denotes the Lebesgue measure on (the Borel σ field of) $[0, 1]$.

Proposition 6.2: *Let $X = [0, 1]$, $I = [1, 4]$; given some $\overline{\mu} \in I$, define $f_{\overline{\mu}}(x) = \overline{\mu}x(1 - x)$ for $x \in X$. Suppose there is a stable periodic orbit. Then for m almost every $x \in [0, 1]$, $\omega(x)$ coincides with this orbit.*

Combining the above two results, we have the following scenario. Suppose we do have a stable periodic orbit. Then there are no other stable periodic orbits. Furthermore, the (unique) stable periodic orbit "attracts" the trajectories from almost every initial state. Thus we can make the qualitative prediction that the asymptotic behavior of the "typical" trajectory will be just like the given stable periodic orbit.

Proposition 6.3: *Let $X = [0, 1]$, $I = [1, 4]$; given some $\overline{\mu} \in I$, define $f_{\overline{\mu}}(x) = \overline{\mu}x(1 - x)$ for $x \in X$. Suppose there is a stable periodic orbit. Then the dynamical system does not have Guckenheimer dependence on initial conditions.*

It is important to note that the above scenario (existence of a stable periodic orbit) is by no means inconsistent with condition (3.1) of the Li-Yorke theorem (and hence with its implications). Let us elaborate on this point following Devaney (1986) and Day and Pianigiani (1991). Consider $\mu = 3.839$, and for this μ, simply write $f(x) = \mu x(1-x)$ for $x \in X$. Choosing $x^* = 0.1498$, it can be checked then that there is $0 < \varepsilon < 0.0001$ such that $f^3(x)$ maps the interval $U \equiv [x^* - \varepsilon, x^* + \varepsilon]$ into itself, and $|[f^3(x)]'| < 1$ for all $x \in U$. Hence, there is $\widehat{x} \in U$ such that $f^3(\widehat{x}) = \widehat{x}$, and $|[f^3(\widehat{x})]'| < 1$. Thus, $\widehat{x}$ is a periodic point of period 3, and it can be checked (by choice of the range of ε) that $f^3(\widehat{x}) = \widehat{x} < f(\widehat{x}) < f^2(\widehat{x})$ so that condition (3.1) of Proposition 3.1 is satisfied. Also, $\widehat{x}$ is a periodic point of period 3 which is *stable*, so that Proposition 3.1 is also applicable. Then we may conclude that the set S of "chaotic" inital states in Proposition 6.3 must be of Lebesgue measure zero. In other words, Li-Yorke chaos exists but is *not* "observed" when $\mu = 3.839$.

7. Comparative Statics and Dynamics

7.1 Bifurcation Theory

Fixed and periodic points formally capture the intuitive idea of a *stationary* state of a dynamical system. In his *Foundations*, Samuelson (1947) noted that (p.513):

"*Stationary* is a descriptive term characterizing the behavior of an economic variable over time; it usually implies constancy, but is occasionally generlized to include behavior periodically repetitive over time."

Bifurcation theory deals with the question of changes in the stationary state of a dynamical system with respect to variations of a parameter that affects the law of motion. Perhaps it is best to introduce the main ideas through an explicit example.

Example 2.1(continued): Consider a class of dynamical systems with a common state space $X = \Re$, and with the laws of motion given by

$$f(x) = x^2 + c \tag{7.1}$$

where 'c' (a real number) is a parameter. Our task is to study the implications of changes in the parameter c. In order to understand stationary states, we have to explore the nature of fixed points of f_c and its iterates f_c^j. To begin with consider the dependence of the fixed points of f_c on the value of c. The fixed points of f_c are obtained by solving

$$f_c(x) = x \tag{7.2}$$

for a given value of c; i.e., solving

$$x^2 - x + c = 0 \tag{7.3}$$

The exact values are given by

$$\begin{aligned} p_+(c) &= (1+\sqrt{1-4c})/2 \\ p_-(c) &= (1-\sqrt{1-4c})/2 \end{aligned} \tag{7.4}$$

Observe that $p_+(c)$ and $p_-(c)$ are real if and only if $1-4c \geq 0$ or $c \leq 1/4$. Thus, when $c > 1/4$, we can say that the dynamical system (with the state space $X = \Re$) has no fixed point when $c = 1/4$, we have

$$p_+\,(1/4) = p_-\,(1/4) = 1/2 \tag{7.5}$$

Now when $c < 1/4$, both $p_+(c)$ and $p_-(c)$ are real and distinct (and $p_+(c) > p_-(c)$).

Thus, as the parameter values decreases through 1/4, the dynamical system undergoes a *bifurcation* from the situation of no fixed piont through a unique fixed point 'splittig into' two.

We can also explore the 'local' dynamics of the system from initial states close to the fixed point(s). Since $f_c'(x) = 2x$ (does not depnd on c), we see that $f_{1/4}'(1/2) = 1$ (so the fixed point 1/2 is 'neutral' when $c = 1/4$). Now from (7.4), we see that:

$$\begin{aligned} f_c'(p_+(c)) &= 1+\sqrt{1-4c} \\ f_c' p_-(c)) &= 1-\sqrt{1-4c} \end{aligned} \tag{7.6}$$

For $c < 1/4$, $f_c'(p_+(c) > 1$ so that the fixed point is repelling.

Now consider $p\_(c)$; of course, $f_c'(p\_(c)) = 1$ when $c = 1/4$. When $c < 1/4$ but sufficiently close to 1/4, $f_c'(p_-(c)) < 1$, so that the fixed point p_- is locally stable or attracting. It will continue to be attracting as long as $|f_c'(p_-(c))| < 1$, i.e.,

$$-1 < f_c'(p\_(c)) < 1$$

or

$$-1 < 1-\sqrt{1-4c} < 1.$$

It follows that $p_-(c)$ is locally stable for all c satisfying

$$-3/4 < c < 1/4$$

when $c = -3/4$, $p_-(c)$ is again 'neutral' (i.e., $|f_c'(p_c)| = 1$) and for $c < -3/4$, $|f_c'(p_-(c))| > 1$ so that $p_-(c)$, too, becomes repelling.

Using a graphical analysis [See Devaney (1992, p. 54-55)] one can show the following:

(i) for $c \leq 1/4$, if the initial state $x > p_+(c)$ or $x < -p_+(c)$ then the trajectory from x tend to infinity;

(ii) for $-3/4 < c < 1/4$ all the trajectories starting from $x \in (-p_+(c), p_+(c))$ tend to the attracting fixed point $p_-(c)$.

As the parameter c decreases through $-3/4$, the fixed point $p_-(c)$ loses it stability property; but more is true. We see a 'period doubling bifurcation': a pair of periodic points is "born". To examine this, consider the equation $f_c^2(x) = x$, i.e.,

$$x^4 + 2cx^2 - x + c^2 + c = 0 \tag{7.7}$$

using the fact that both p_+ and p_- are solutions to (7.7), we see that there are two other roots given by

$$\begin{aligned} q_+(c) &= (-1 + \sqrt{-4c-3})/2 \\ q_-(c) &= (-1 - \sqrt{-4c-3})/2 \end{aligned} \tag{7.8}$$

Clearly, $q_+(c)$ and $q_-(c)$ are real if an only if $c \leq -3/4$.

Of course, when $c = -3/4$, $q_+(c) = q_-(c) = -1/2 = p_-(c)$. Furthermore, for $-5/4 < c < -3/4$, the periodic points are locally stable.

To summarize:

a change in the parameter may affect the number as well as the local stability properties of fixed and periodic points of a family of dynamical systems.

The above indicates that bifurcations occur near non-hyperbolic fixed and periodic points. This is indeed the only place where bifurcations of fixed points occur, as the follwoing result demonstrates.

Proposition 7.1: *Let f_λ be a one-parameter family of functions and suppose that $f_{\lambda_0}(x_0) = x_0$ and $f_{\lambda_0}'(x_0) \neq 1$. Then there are intervals I about x_0 and N about λ_0 and a smooth function $p : N \to I$ such that $p(\lambda_0) = x_0$ and $f_\lambda(p(\lambda)) = p(\lambda)$. Moreover, f_λ has no other fixed points in I.*

Proof: Consider the function defined by $G(x, \lambda) = f_\lambda(x) - x$. By hypothesis, $G(x_0, \lambda_0) = 0$ and

$$\frac{\partial G}{\partial x}(x_0, \lambda_0) = f'_{\lambda_0}(x_0) - 1 \neq 0$$

By the Implicit Function Theorem, there are intervals I about x_0 and N about λ_0, and a smooth function $p : N \rightarrow I$ such that $p(\lambda_0) = x_0$ and $G(p(\lambda), \lambda) = 0$ for all $\lambda \in N$. Moreover, $G(x, \lambda) \neq 0$ unless $x = p(\lambda)$. This concludes the proof.

Remark: For theoretical simplicity, it is often convenient to assume that the fixed point set of f_λ is statioanry as λ is varied. The previous result allows us to make this assumption. Suppose that f_λ and $f_\lambda(p(\lambda)) = p(\lambda)$ are as in Proposition 7.1. Consider the new function

$$g_\lambda(z) = f_\lambda(z + p(\lambda)) - p(\lambda).$$

Clearly, $g_\lambda(0) = f_\lambda(p(\lambda)) - p(\lambda) = 0$ for all λ, so 0 is always fixed. Moroever, g_λ is topologically conjugate [Devaney (1986, p. 47)] to f_λ via the simple map $h_\lambda(x) = x - p(x)$. Hence the dynamics of f_λ and g_λ agree, but g_λ is simpler to handle since fixed point remains stationary at 0 as λ varies.

The above proposition (as well as all that follow) obviously hold for periodic points by replacing f with f^n.

We now turn to the general setting of bifurcation theory.

Proposition 7.2 (The saddle-node bifurcation): *Suppose that*

(i) $f_{\lambda_0}(0) = 0$
(ii) $f'_{\lambda_0}(0) = 1$
(iii) $f''_{\lambda_0}(0) \neq 0$
(iv) $\frac{\partial f_\lambda}{\partial \lambda}\Big|_{\lambda=\lambda_0} \neq 0$

Then there exists an interval I about 0 and a smooth function $p : I \rightarrow R$ such that

$$f_{p(x)}(x) = x.$$

Moreover, $p'(0) = 0$ and $p''(0) \neq 0$.

Remark: The signs of $f''_\lambda(0)$ and

$$\left.\frac{\partial f_\lambda}{\partial \lambda}\right|_{\lambda=\lambda_0}$$

determine the "direction" of the bifurcation.

Proof: Define $G(x,\lambda) = f_\lambda(x) - x$. Note that $G(x,\lambda) = 0$ implies that f_λ has a fixed point at x. We will apply the Implicit Function Theorem to G.

Note that $G(0,\lambda_0) = 0$ and that

$$\frac{\partial G}{\partial \lambda}(0,\lambda_0) = \left.\frac{\partial f_\lambda}{\partial \lambda}\right|_{\lambda=\lambda_0(0)} \neq 0.$$

Hence there exists a smooth function $p(x)$ satisfying $G(x,p(x)) = 0$. From the chain rule, we have

$$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial \lambda} p'(x) = 0.$$

Therefore

$$p'(x) = \frac{-\frac{\partial G}{\partial x}}{\frac{\partial G}{\partial \lambda}}.$$

Verify that $p'(0) = 0$ and $p''(0) \neq 0$ [see Devaney (1986)]. Q.E.D.

Proposition 7.3 (Period-doubling bifurcation): *Suppose*

(i) $f_\lambda(0) = 0$ *for all* λ *in an interval about* λ_0

(ii) $f'_{\lambda_0}(0) = -1$

(iii) $f'''_{\lambda_0}(0) \neq 0$

(iv) $\left.\frac{\partial (f^2_\lambda)'}{\partial \lambda}\right|_{\lambda=\lambda_0}(0) \neq 0$

Then there is an interval I *about* 0 *and a function* $p : I \to R$ *such that*

$$f_{p(x)}(x) \neq x$$

but

$$f^2_{p(x)}(x) = x.$$

Proof: For this proof we define $G(x, \lambda) = f_\lambda^2(x) - x$. Also set

$$H(x, \lambda) = \begin{cases} \frac{G(x,\lambda)}{x} & x \neq 0 \\ \frac{\partial G}{\partial x}(0, \lambda) & x = 0. \end{cases}$$

One checks easily that H is smooth and satisfies

$$\frac{\partial H}{\partial x}(0, \lambda_0) = \frac{\partial^2 G}{\partial x^2}(0, \lambda_0)$$

$$\frac{\partial^2 H}{\partial x^2}(0, \lambda_0) = \frac{\partial^3 G}{\partial x^3}(0, \lambda_0).$$

We now apply the Implicit Function Theorem to H. Note that

$$\begin{aligned} H(0, \lambda_0) &= \frac{\partial G}{\partial x}(0, \lambda_0) \\ &= (f_{\lambda_0}^2)'(0) - 1 \\ &= f_{\lambda_0}'(0) \bullet f_\lambda'(0) - 1 \\ &= 0. \end{aligned}$$

We have by assumption that

$$\begin{aligned} \frac{\partial H}{\partial \lambda}(0, \lambda_0) &= \left.\frac{\partial}{\partial \lambda}\right|_{\lambda=\lambda_0} (f_\lambda^2)'(0) - 1 \\ &= \frac{\partial (f_\lambda^2)'}{\partial \lambda}(0) \\ &\neq 0. \end{aligned}$$

Hence there is a smooth function $p(x)$ defined on a neighborhood of 0 and satisfying $p(0) = \lambda_0$ and $H(x, p(x)) = 0$. In particular,

$$\frac{1}{x} G(x, p(x)) = 0$$

or $x \neq 0$ and it follows that x is a period two point for $f_{p(x)}$.

As above, we compute

$$p'(0) = \frac{-\frac{\partial H}{\partial x}(0, \lambda_0)}{\frac{\partial H}{\partial \lambda}(0, \lambda_0)} = 0 \text{ and } p''(0) \neq 0 \qquad \text{Q.E.D.}$$

7.2 Robustness

Let $X = [a, b]$ and suppose that a continuous function f satisfies the Li-Yorke condition (3.1) with strict inequality throughout, i.e., suppose that there are points a, b, c, d such that

$$\begin{aligned} d &= f(c) < a < b = f(a) < c = f(b) \\ &\text{or} \\ d &= (f(c)) > a > b = f(a) > c = f(b) \end{aligned} \tag{7.9}$$

Consider $C(X)$ the space of all bounded continuous real valued functions on X. Let $||f|| = \max_{x \in X} |f(x)|$. The remarkable conclusions of Theorem 3.1 then hold with respect to the dynamical system (X, f). But the complexity is "robust" in the following sense.

Proposition 7.4: *Suppose that $X = [a, b]$ and f satisfies (7.9). In addition, suppose that $a < r(X, f) < R(X, f) < b$, where $R(X, f)$ and $r(X, f)$ are respectively the maximum and the minimum of f on the interval X. Then there exists some $\varepsilon > 0$ such that for all g with $||g - f|| < \varepsilon$, the conclusions of Proposition 3.1 hold with g in place of f.*

Proof: see Bala and Majumdar [1992, Corollary 3.1].

Finally, consider the parameteric family $\mathcal{F}$ of maps $f_\mu : [0, 1] \times [3, 4] \to [0, 1]$ defined as:

$$\mathcal{F} = \{f_\mu(x) = \mu x(1 - x), x \in [0, 1], \mu \in [2, 4]$$

The following fundamental theorem of Jakobson (1981) throws light on the issue of robust ergodic chaos.

Proposition 7.5: *There is a set $M \subset [3, 4]$ with $m(M) > 0$ such that if $X = [0, 1]$, $\mu \in M$, and $f_\mu \equiv \mu x(1 - x)$, then (X, f_μ) exhibits ergodic chaos.*